# Formality properties: generalizations and applications

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- For a "formal" simply connected space, its rational homotopy type is determined by its cohomology algebra over $\mathbb{Q}$.

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  equivalently, there is a sequence of zig-zag quasi-isomorphisms

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**Corollary** [Sullivan 77] Compact Kähler manifolds are formal over $\mathbb{Q}$.

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**Theorem (Santos–Navarro–Pascual–Roig 05)**

*Let $\mathbb{Q} \subset \mathbb{K}$ be a field extension, and $P$ be a dg operad over $\mathbb{Q}$ with homology of finite type. $P$ is formal if and only if $P \otimes \mathbb{K}$ is formal.*

# Partial formality

- A CDGA morphism $f\colon A \to B$ is an *$i$-quasi-isomorphism* if

$$f^*\colon H^j(A) \to H^j(B)$$

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**Theorem (Suciu–W.)**

*Let $\mathbb{Q} \subset \mathbb{K}$ be a field extension, and $X$ be a connected space with finite Betti numbers $b_1(X), \dots, b_{i+1}(X)$. Then $X$ is i-formal over $\mathbb{Q}$ if and only if $X$ is i-formal over $\mathbb{K}$.*

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An obstruction to formality is provided by non-vanishing higher Massey products.

# Graded Lie algebras

Let $G$ be a finitely generated group.

- The *lower central series* of $G$: $\Gamma_1 G = G$, $\Gamma_2 G = [G, G]$, $\Gamma_{k+1} G = [\Gamma_k G, G]$, $k \geq 1$.

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# Malcev Lie algebra

Let $G$ be a finitely generated group.

- There exists a tower of nilpotent Lie algebras [Malcev 51]

$$\mathfrak{L}((G/\Gamma_2 G)\otimes \mathbb{Q}) \twoheadleftarrow \mathfrak{L}((G/\Gamma_3 G)\otimes \mathbb{Q}) \twoheadleftarrow \mathfrak{L}((G/\Gamma_4 G)\otimes \mathbb{Q}) \twoheadleftarrow$$

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- The universal enveloping algebra of $\mathfrak{m}(G;\mathbb{Q})$ is isomorphic to $\widehat{\mathbb{Q}G}$. [Quillen 69]
- Let $\mathcal{M}(G,1)$ be the 1-minimal model of $K(G,1)$.
  These is a one to one corresponding between $\mathcal{M}(G,1)$ and the Malcev Lie algebra $\mathfrak{m}(G;\mathbb{Q})$. [Sullivan 77, Cenkl–Porter 81]

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\text{filtered-formal} \downarrow & & \downarrow \text{graded-formal} \\
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- formal $\Longrightarrow$ $i$-formal $\Longrightarrow$ 1-formal $\Longleftrightarrow$ $\begin{array}{c}\text{graded-formal}\\ + \\ \text{filtered-formal.}\end{array}$

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- $G$ has a Taylor expansion $\Longleftrightarrow$ $G$ is filtered-formal.
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## Remark

- The filtered formality of finite-dimensional, nilpotent Lie algebras has been studied under many different names: 'Carnot', 'naturally graded', 'homogeneous' and 'quasi-cyclic'. In this special case, the above theorem was proved by Cornulier (14).
- A finitely generated, torsion-free, 2-step nilpotent group is filtered-formal. [Suciu-W.]
- Recently, Bar-Natan has explored the "Taylor expansion" of $G$, which is a map $E\colon G \to \widehat{\mathrm{gr}}(\mathbb{Q}G)$ satisfing some properties.
- $G$ has a Taylor expansion $\Longleftrightarrow$ $G$ is filtered-formal.
  $G$ has a quadratic Taylor expansion $\Longleftrightarrow$ $G$ is 1-formal.
- A map $T\colon F_n \to \mathbb{Q}\langle\langle z_1, \cdots, z_n \rangle\rangle$ defined by $T(x_i) = \exp(z_i)$ is a Taylor expansion of $F_n$.

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1. *If $G$ is graded-formal, then $K$ is also graded-formal.*
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## Proposition (Suciu–W.)

*Let $G_1$ and $G_2$ be two finitely generated groups. The following conditions are equivalent.*

1. *$G_1$ and $G_2$ are graded-formal (respectively, filtered-formal, or 1-formal).*
2. *$G_1 * G_2$ is graded-formal (respectively, filtered-formal, or 1-formal).*
3. *$G_1 \times G_2$ is graded-formal (respectively, filtered-formal, or 1-formal).*

# An obstruction to 1-formality: resonance varieties

- Suppose $A^* := H^*(G, \mathbb{C})$ has finite dimension in each degree.

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**Theorem (Dimca–Papadima–Suciu 09)**

*If $G$ is 1-formal, $\mathcal{R}^1_k(G, \mathbb{C})$ is a union of rationally defined linear subspaces of $H^1(G, \mathbb{C})$.*

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- The *pure virtual braid groups* of $vP_n$ has a presentation [Bardakov (04)] with generators $x_{ij}$ for $1 \leq i \neq j \leq n$, subject to the relations

$$\begin{aligned} &x_{ij}x_{ik}x_{jk} = x_{jk}x_{ik}x_{ij}, && \text{for } i,j,k \text{ distinct,} \\ &[x_{ij}, x_{st}] = 1, && \text{for } i,j,s,t \text{ distinct.} \end{aligned}$$

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- They also showed that $vP_n$ and $vP_n^+$ are *graded-formal* (with the work of P. Lee (13) ) and computed the cohomology algebras of these groups.

# (Non-)formality of pure virtual braid groups

**Theorem (Suciu–W.)**

*The pure virtual braid groups $vP_n$ and $vP_n^+$ are 1-formal if and only if $n \leq 3$.*

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Sketch of proof:

**Lemma**

*There are split monomorphisms*

$$\begin{array}{ccccccccccc}
vP_2^+ & \hookrightarrow & vP_3^+ & \hookrightarrow & vP_4^+ & \hookrightarrow & vP_5^+ & \hookrightarrow & vP_6^+ & \hookrightarrow & \dots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
vP_2 & \hookrightarrow & vP_3 & \hookrightarrow & vP_4 & \hookrightarrow & vP_5 & \hookrightarrow & vP_6 & \hookrightarrow & \dots
\end{array}$$

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*The group $vP_4^+$ is not 1-formal.*

Proof: The first resonance variety $\mathcal{R}_1(vP_4^+, \mathbb{C})$ is the subvariety of $\mathbb{C}^6$ given by the equations

$$x_{12}x_{24}(x_{13}+x_{23}) + x_{13}x_{34}(x_{12}-x_{23}) - x_{24}x_{34}(x_{12}+x_{13}) = 0,$$
$$x_{12}x_{23}(x_{14}+x_{24}) + x_{12}x_{34}(x_{23}-x_{14}) + x_{14}x_{34}(x_{23}+x_{24}) = 0,$$
$$x_{13}x_{23}(x_{14}+x_{24}) + x_{14}x_{24}(x_{13}+x_{23}) + x_{34}(x_{13}x_{23}-x_{14}x_{24}) = 0,$$
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\end{aligned}
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$\Rightarrow$ The group $vP_4^+$ is not 1-formal. □

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*Thank You!*