Formality properties, resonance varieties and Chen ranks

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Overview

Graded Lie algebras

- G : a finitely generated group.
- $\mathbb{k}$: a field of characteristic 0.
- The *lower central series* of G: $\Gamma_1 G = G$, $\Gamma_{k+1} G = [\Gamma_k G, G]$, $k \geq 1$.
- The *associated graded Lie algebra* of a group G is defined by

$$\operatorname{gr}(G;\mathbb{k}) := \bigoplus_{k\geq 1} (\Gamma_k G/\Gamma_{k+1} G) \otimes_{\mathbb{Z}} \mathbb{k}. \tag{1}$$

- The *holonomy Lie algebra* of a group G is defined to be

$$\mathfrak{h}(G;\mathbb{k}) := \operatorname{Lie}(H_1(G;\mathbb{k}))/\langle \operatorname{im}(\partial_G)\rangle. \tag{2}$$

  Here, ∂_G is the dual of $H^1(G;\mathbb{k}) \wedge H^1(G;\mathbb{k}) \xrightarrow{\cup} H^2(G;\mathbb{k})$.
- There exists an epimorphism $\Phi_G : \mathfrak{h}(G;\mathbb{k}) \twoheadrightarrow \operatorname{gr}(G;\mathbb{k})$. [Lambe 86]

Magnus expansion

- Suppose G has a finite presentation $\langle x_1, \ldots, x_n \mid r_1, \ldots, r_m \rangle$.
- The *Magnus expansion* $M\colon \Bbbk F \to \Bbbk\langle\langle x_1, \ldots, x_n \rangle\rangle$, is a ring homomorphism defined by $M(x_i) = 1 + x_i$ and $M(x_i^{-1}) = 1 - x_i + x_i^2 - x_i^3 + \cdots$.

Theorem (Fenn-Sjerve 87, Matei-Suciu 98)

Suppose G is a commutator-relators group, such that $H_2(G)$ is free abelian. The cup-product $\cup\colon H^1(G) \wedge H^1(G) \to H^2(G)$ is given by

$$u_i \cup u_j = \sum_{k=1}^{m} M(r_k)_{i,j} \beta_k.$$

Proposition (Papadima-Suciu 04)

If G is a commutator-relators group, then

$$\mathfrak{h}(G; \Bbbk) = \mathrm{Lie}(x_1, \ldots, x_n) / \langle M_2(r_1), \ldots, M_2(r_m) \rangle.$$

Generalization

- The *Magnus expansions* of a group G is the composition

$$\kappa: \quad \Bbbk F \xrightarrow{M} \Bbbk\langle\langle x_1, \ldots, x_n \rangle\rangle \xrightarrow{\widehat{\pi}} \Bbbk\langle\langle y_1, \ldots, y_b \rangle\rangle,$$

  where $b = \dim H_1(G; \Bbbk)$ and $\widehat{\pi}$ is induced by $\pi\colon H_1(F; \Bbbk) \to H_1(G; \Bbbk)$.
- In particular, if G is a commutator-relators group, then $\widehat{\pi}$ is identity.
- A group G has an *echelon presentation* $\langle x_1, \ldots, x_n \mid w_1, \ldots, w_m \rangle$, if the augmented Jocobian matrix of Fox derivative $(M(w_k)_i)$ is in row-echelon form.
- $\kappa_2(r)$: the degree 2 homogeneous part of $\kappa(r)$.
- $\kappa(r)_{i,j}$: the coefficient of $y_i y_j$ in $\kappa(r)$.

Theorem (Suciu-W. 15)

Let G be a group with a presentation $\langle x_1, \dots, x_n \mid r_1, \dots, r_m \rangle$.
Let K_G be the 2-complex associated to the presentation.

1. *There exists a group $\widetilde{G}$ with echelon presentation $\langle x_1, \dots, x_n \mid w_1, \dots, w_m \rangle$ such that*

$$\mathfrak{h}(G;\Bbbk) \cong \mathfrak{h}(\widetilde{G};\Bbbk) \text{ and } H^{\leq 2}(K_G;\Bbbk) \cong H^{\leq 2}(K_{\widetilde{G}};\Bbbk).$$

2. *The cup-product map $\cup\colon H^1(K_G;\Bbbk) \wedge H^1(K_G;\Bbbk) \to H^2(K_G;\Bbbk)$ is given by*

$$u_i \cup u_j = \sum_{k=n-b+1}^{m} \kappa(w_k)_{i,j} \beta_k.$$

3. *There exists an isomorphism of graded Lie algebras*

$$\mathfrak{h}(G;\Bbbk) \xrightarrow{\cong} \mathrm{Lie}(y_1, \dots, y_b)/\mathrm{ideal}(\kappa_2(w_{n-b+1}), \dots, \kappa_2(w_m)) .$$

Malcev Lie algebra

- The tower of nilpotent Lie groups

$$\cdots \longrightarrow (G/\Gamma_4 G)\otimes \Bbbk \longrightarrow (G/\Gamma_3 G)\otimes \Bbbk \longrightarrow (G/\Gamma_2 G)\otimes \Bbbk$$

is an inverse limit system. The pronilpotent Lie algebra defined by

$$\mathfrak{m}(G;\Bbbk) = \varprojlim_k (\mathfrak{L}((G/\Gamma_k G)\otimes \Bbbk)), \tag{3}$$

is called the *Malcev Lie algebra* of G (over $\Bbbk$).

- $A := \Bbbk G$ with a natural Hopf algebra structure.
- $\mathfrak{p}(G;\Bbbk) := \{$ all primitive elements of $\hat{A}$ $\}$.

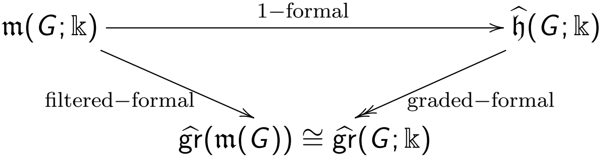
Theorem (Quillen)

1. *There is a filtered Lie algebra isomorphism* $\mathfrak{p}(G;\Bbbk) \to \mathfrak{m}(G;\Bbbk)$.
2. *There is a graded Lie algebra isomorphism* $\mathrm{gr}(G;\Bbbk) \to \mathrm{gr}(\mathfrak{m}(G;\Bbbk))$.

Formality

- A group G is *1-formal*, if there exists a cdga homomorphism $\mathcal{M}(G) \to H^*(G;\mathbb{Q})$, inducing an isomorphism in cohomology of degree 1 and a monomorphism in degree 2.
- A group G is 1-formal iff $\mathfrak{m}(G;\Bbbk) \cong \widehat{\mathfrak{h}}(G;\Bbbk)$. [Sullivan 77]
- A group G is *graded-formal*, if $\Phi_G \colon \mathfrak{h}(G;\Bbbk) \twoheadrightarrow \operatorname{gr}(G;\Bbbk)$ is an isomorphism of graded Lie algebras.
- A group G is *filtered-formal*, if there is a filtered Lie algebra isomorphism $\mathfrak{m}(G;\Bbbk) \cong \widehat{\operatorname{gr}}(G;\Bbbk)$, which induces the identity on associated graded Lie algebras.

  - 1-formal $\Longleftrightarrow$ graded-formal + filtered-formal.



See my poster tomorrow for more properties!

The resonance varieties

- G : finitely generated group.
- Suppose $A := H^*(G, \mathbb{C})$ has finite dimension in each degree.
- For each $a \in A^1$, we have $a^2 = 0$.
- Define a cochain complex of finite-dimensional $\mathbb{C}$-vector spaces,

$$(A, a) : A^0 \xrightarrow{a\cup -} A^1 \xrightarrow{a\cup -} A^2 \xrightarrow{a\cup -} \cdots ,$$

with differentials given by left-multiplication by a.

Definition

The *resonance varieties* of G are the homogeneous subvarieties of A^1

$$\mathcal{R}^i_d(G, \mathbb{C}) = \{a \in A^1 \mid \dim_{\mathbb{C}} H^i(A; a) \geq d\},$$

defined for all integers $i \geq 1$ and $d \geq 1$.

- $\mathcal{R}^1_1(\mathbb{Z}^n, \mathbb{C}) = \{0\}$; $\mathcal{R}^1_1(\pi_1(\Sigma_g), \mathbb{C}) = \mathbb{C}^{2g}$, $g \geq 2$.

Alexander invariants

- *Alexander invariant* is the $\mathbb{Z}[G_{\mathrm{ab}}]$-module $B(G) = G'/G''$, where $G' = [G, G]$ and $G'' = [G', G']$ are the 1st and 2nd derived subgroups.
- The $\mathbb{Z}[G_{\mathrm{ab}}]$-module structure on $B(G)$ is determined by the extension

$$0 \to G'/G'' \to G/G'' \to G/G' \to 0.$$

  with G/G' acting on the cosets of G'' via conjugation: $gG' \cdot hG'' = ghg^{-1}G''$, for $g \in G$, $h \in G'$.
- Let $\mathfrak{g}$ be a finitely generated graded Lie algebra. The *infinitesimal Alexander invariant* of $\mathfrak{g}$ is the graded S-module $\mathfrak{B}(\mathfrak{g}) := \mathfrak{g}'/\mathfrak{g}''$. Here, S is universal enveloping algebra of $\mathfrak{g}/\mathfrak{g}'$.
- If G is a commutator-relators group, there is an isomorphism $B^{lin}(G) \otimes \Bbbk \cong \mathfrak{B}(\mathfrak{h}(G; \Bbbk))$. [Papadima-Suciu 04]

Chen Lie algebras

- The *Chen Lie algebra* of a group G is defined to be

$$\mathrm{gr}(G/G'';\Bbbk) := \bigoplus_{k\geq 1}(\Gamma_k(G/G'')/\Gamma_{k+1}(G/G'')) \otimes_{\mathbb{Z}} \Bbbk.$$

- The quotient map $h\colon G \twoheadrightarrow G/G''$ induces $\mathrm{gr}(G;\Bbbk) \twoheadrightarrow \mathrm{gr}(G/G'';\Bbbk)$.
- The *LCS ranks* of G are defined as $\phi_k(G) := \mathrm{rank}(\mathrm{gr}_k(G;\Bbbk))$.
- The *Chen ranks* of G are defined as $\theta_k(G) := \mathrm{rank}(\mathrm{gr}_k(G/G'';\Bbbk))$.
- $\theta_k(G) = \phi_k(G)$ for $k \leq 3$.
- $\theta_k(F_n) = (k-1)\binom{n+k-2}{k}$, $k \geq 2$. [Chen 51]

Hilbert series and Chen ranks

- $I := \ker \epsilon \colon \mathbb{Z}[G_{\mathrm{ab}}] \to \mathbb{Z}$.
- The module $B(G)$ has an I-adic filtration $\{I^k B(G)\}_{k\geq 0}$.
- $\mathrm{gr}(B(G)) = \bigoplus_{k\geq 0} I^k B(G)/I^{k+1}B(G)$ is a graded $\mathrm{gr}(\mathbb{Z}[G_{\mathrm{ab}}])$-module.

Proposition (Massey 80)

For each $k \geq 2$, there exists an isomorphism

$$\mathrm{gr}_k(G/G'') \cong \mathrm{gr}_{k-2}(B(G)).$$

Corollary

$$\mathrm{Hilb}(B(G)\otimes \Bbbk, t) = \sum_{k\geq 0} \theta_{k+2}(G) t^k.$$

Theorem (Labute 08, Suciu-W. 15)

For each $i \geq 2$, the quotient map $G \twoheadrightarrow G/G^{(i)}$ induces a natural epimorphism of graded $\Bbbk$-Lie algebras,

$$\Psi_G^{(i)}\colon \operatorname{gr}(G;\Bbbk)/\operatorname{gr}(G;\Bbbk)^{(i)} \longrightarrow\!\!\!\!\rightarrow \operatorname{gr}(G/G^{(i)};\Bbbk)\,.$$

Moreover, if G is a filtered-formal group, then each solvable quotient $G/G^{(i)}$ is also filtered-formal, and the map $\Psi_G^{(i)}$ is an isomorphism.

Corollary (Papadima-Suciu 04)

If G is a 1-formal group, then $\mathfrak{h}(G;\Bbbk)/\mathfrak{h}(G;\Bbbk)^{(i)} \cong \operatorname{gr}(G/G^{(i)};\Bbbk)$.

Corollary (Papadima-Suciu 04)

If G is a 1-formal group, then

$$\operatorname{Hilb}(\mathfrak{B}(\mathfrak{h}(G;\Bbbk)),t) = \sum_{k\geq 0} \theta_{k+2}(G)t^k.$$

McCool groups (pure welded braid groups) (group of loops)

- The McCool group $P\Sigma_n$ is the group of basis-conjugating automorphisms, which is a subgroup of
$$IA_n := \ker(\mathrm{Aut}(F_n) \twoheadrightarrow \mathrm{GL}_n(\mathbb{Z})).$$
- The McCool groups $P\Sigma_n$ has a presentation [McCool (86)] with generators: x_{ij}, for $1 \leq i \neq j \leq n$ and relations: $x_{ij}x_{ik}x_{jk} = x_{jk}x_{ik}x_{ij}$; $[x_{ij}, x_{kl}] = 1$; $[x_{ij}, x_{kj}] = 1$, for i, j, k, l distinct.
- $H^*(P\Sigma_n, \mathbb{C})$ was computed by Jensen, McCammond, and Meier (06).

Theorem (D.Cohen 09)

The first resonance variety of McCool group $P\Sigma_n$ is

$$\mathcal{R}_1^1(P\Sigma_n, \mathbb{C}) = \bigcup_{1\leq i<j\leq n} C_{ij} \cup \bigcup_{1\leq i<j<k\leq n} C_{ijk},$$

where $C_{ij} = \mathbb{C}^2$ and $C_{ijk} = \mathbb{C}^3$.

Upper McCool groups

- The upper McCool group $P\Sigma_n^+$ is the subgroup of $P\Sigma_n$ generated by the x_{ij} for $1 \leq i < j \leq n$.
- $P\Sigma_n$ and $P\Sigma_n^+$ are 1-formal.[Berceanu-Papadima 09]
- F. Cohen, Pakhianathan, Vershinin, and Wu (07) computed the cohomology ring $H^*(P\Sigma_n^+;\mathbb{Z})$. The LCS ranks $\phi_k(P\Sigma_n^+) = \phi_k(P_n)$ and the Betti numbers $b_k(P\Sigma_n^+) = b_k(P_n)$, where P_n is the pure braid group. They ask a question: are $P\Sigma_n^+$ and P_n isomorphic for $n \geq 4$?
- For $n = 4$, the question was answered by Bardakov and Mikhailov (08) using Alexander polynomials. (There is a gap in their proof.)

Theorem (Suciu, W. 15)

The Chen ranks θ_k of $P\Sigma_n^+$ are given by $\theta_1 = \binom{n}{2}, \theta_2 = \binom{n}{3}, \theta_3 = 2\binom{n+1}{4}$,

$$\theta_k = \binom{n+k-2}{k+1} + \theta_{k-1} = \sum_{i=3}^{k} \binom{n+i-2}{i+1} + \binom{n+1}{4}, \; k \geq 4.$$

Corollary

The pure braid group P_n, the upper McCool group $P\Sigma_n^+$, and the product group $\Pi_n := \prod_{i=1}^{n-1} F_i$ are not isomorphic for $n \geq 4$.

Proof:

$$\theta_4(P_n) = 3\binom{n+1}{4}, \theta_4(P\Sigma_n^+) = 2\binom{n+1}{4} + \binom{n+2}{5}, \theta_4(\Pi_n) = 3\binom{n+2}{5}.$$

The Chen ranks of P_n and Π_n were computed by D. Cohen and Suciu (95).

Theorem (Suciu, W. 15)

The first resonance variety of the upper McCool group $P\Sigma_n^+$ is

$$\mathcal{R}_1^1(P\Sigma_n^+, \mathbb{C}) = \bigcup_{n \geq i > j \geq 2} C_{i,j},$$

where $C_{i,j} = \mathbb{C}^j$.

Remark (Chen ranks conjecture, Suciu 01, Schenck-Suciu 04, D. Cohen-Schenck 14)

Let c_n be the number of n-dimensional components of $\mathcal{R}_1^1(G)$.

$$\theta_k(G) = \sum_{n\geq 2} c_m \cdot \theta_k(F_n), \quad \text{for} \quad k \gg 1.$$

This formula is true if G is a 1-formal, commutator-relators group, such that the resonance variety $\mathcal{R}_1^1(G)$ is 0-isotropic, projectively disjoint, and reduced as a scheme.

Examples satisfying these conditions include arrangement groups and McCool groups. However, the upper McCool groups $P\Sigma_n^+$ do not satisfy this formula for $n \geq 4$.

Thank You!