Formality properties of finitely generated groups

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Introduction

We explore the graded-formality and filtered-formality properties of a finitely-generated group, which are weaker properties than 1-formality in rational homotopy theory. We explain how these notions behave with respect to split injections, coproducts, and direct products, and how they are inherited by solvable and nilpotent quotients by studying the various Lie algebras attached to such a group, including the associated graded Lie algebra, the holonomy Lie algebra, and the Malcev Lie algebra. We also provide various examples to explain these properties.

Backgrounds and Definitions

- G : a finitely generated group.
- $\Bbbk \supset \mathbb{Q}$: a field of characteristic 0.
- $\{\Gamma_k G\}_{k\geq 1}$: the *lower central series* G, defined inductively by
$$\begin{cases}\Gamma_1 G = G \\ \Gamma_{k+1} G = [\Gamma_k G, G],\ k \geq 1.\end{cases} \tag{1}$$
- The *associated graded Lie algebra* of a group G is defined by
$$\operatorname{gr}(G;\Bbbk) := \bigoplus_{k\geq 1} (\Gamma_k G/\Gamma_{k+1} G) \otimes_{\mathbb{Z}} \Bbbk, \tag{2}$$
- The *holonomy Lie algebra* of a group G is defined to be
$$\mathfrak{h}(G;\Bbbk) := \operatorname{Lie}(H_1(G;\Bbbk))/\langle \operatorname{im}(\partial_G)\rangle. \tag{3}$$
Here, ∂_G is the dual of the cup product.

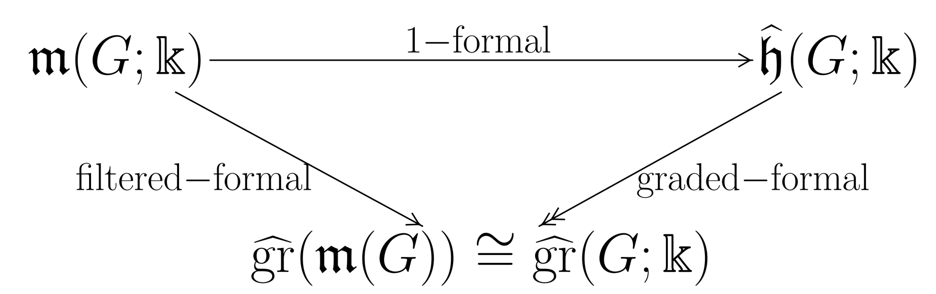
• For a finitely-presented group, we give an explicit algorithm for computing the holonomy Lie algebra, using a Magnus expansion method [1].

- The tower of nilpotent Lie groups
$$\cdots \to (G/\Gamma_4 G)\otimes \Bbbk \to (G/\Gamma_3 G)\otimes \Bbbk \to (G/\Gamma_2 G)\otimes \Bbbk$$
is an inverse limit system. The pronilpotent Lie algebra defined by
$$\mathfrak{m}(G;\Bbbk) = \varprojlim_k (\mathfrak{L}((G/\Gamma_k G)\otimes \Bbbk)), \tag{4}$$
is called the *Malcev Lie algebra* of G (over $\Bbbk$).

Formality Properties

- A group G is called *1-formal* if there exists a cdga morphism $\mathscr{M}(G) \to H^*(G;\mathbb{Q})$ inducing an isomorphism in cohomology of degree 1 and a monomorphism in degree 2, where $\mathscr{M}(G)$ is the 1-minimal model of G.
- A group G is *graded-formal* if the canonical projection $\Phi_G\colon \mathfrak{h}(G;\Bbbk) \twoheadrightarrow \operatorname{gr}(G;\Bbbk)$ is an isomorphism of graded Lie algebras.
- A group G is called *filtered-formal* if there is a filtered Lie algebra isomorphism $\mathfrak{m}(G;\Bbbk) \cong \widehat{\operatorname{gr}}(G;\Bbbk)$ which induces the identity on associated graded Lie algebras. Here, $\widehat{}$ means completion.

• 1-formal $\Longleftrightarrow$ graded-formal + filtered-formal.



Propagation Results

Theorem 1. (Split injections)
Let G be a finitely generated group, and let $K \leq G$ be a subgroup. Suppose there is a split monomorphism $\iota\colon K \to G$. Then:

1. If G is graded-formal, then K is graded-formal.
2. If G is filtered-formal, then K is filtered-formal.
3. If G is 1-formal, then K is also 1-formal.

Theorem 2. (Products & coproducts)
Let G_1 and G_2 be two finitely generated groups. The following conditions are equivalent.

1. G_1 and G_2 are graded-formal (respectively, filtered-formal, or 1-formal).
2. $G_1 * G_2$ is graded-formal (respectively, filtered-formal, or 1-formal).
3. $G_1 \times G_2$ is graded-formal (respectively, filtered-formal, or 1-formal).

Applications

Theorem 3. (Solvable quotients)
Let G be a finitely generated group. For each $i \geq 2$, the quotient map $G \twoheadrightarrow G/G^{(i)}$ induces a natural epimorphism of graded $\Bbbk$-Lie algebras,
$$\Psi_G^{(i)}\colon \operatorname{gr}(G;\Bbbk)/\operatorname{gr}(G;\Bbbk)^{(i)} \twoheadrightarrow \operatorname{gr}(G/G^{(i)};\Bbbk). \tag{5}$$
Moreover,

1. If G is a filtered-formal group, then each solvable quotient $G/G^{(i)}$ is also filtered-formal, and the map $\Psi_G^{(i)}$ is an isomorphism.
2. If G is a 1-formal group, then $\mathfrak{h}(G;\Bbbk)/\mathfrak{h}(G;\Bbbk)^{(i)} \cong \operatorname{gr}(G/G^{(i)};\Bbbk)$.

Theorem 4. (Nilpotent quotients)
Let G be a finitely generated group.

1. If G is filtered-formal, then all the nilpotent quotients $G/\Gamma_k(G)$ are also filtered-formal.
2. Suppose G is a torsion-free, 2-step nilpotent group with torsion-free abelianization. Then G is filtered-formal.
3. Suppose G is a torsion-free, filtered-formal, nilpotent group. Then the universal enveloping algebra $U(\operatorname{gr}(G;\Bbbk))$ is Koszul if and only if G is abelian.

Pure virtual braid groups

The pure virtual braid group vP_n is a generalization of the classical pure braid groups, with a class of subgroups vP_n^+. In algebra, they are called quasi-triangular groups and triangular groups.

Theorem 5. ([2]) The pure virtual braid groups vP_n and vP_n^+ are 1-formal if and only if $n \leq 3$.

Sketch of Proof: (Using Theorem 1)

- There are split monomorphisms
$$\begin{array}{ccccccccccc} vP_2^+ & \to & vP_3^+ & \to & vP_4^+ & \to & vP_5^+ & \to & vP_6^+ & \to & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ vP_2 & \to & vP_3 & \to & vP_4 & \to & vP_5 & \to & vP_6 & \to & \cdots \end{array}$$
- The group vP_3 is 1-formal.
- The first resonance variety of vP_4^+ is not linear.
- The group vP_4^+ is not 1-formal. (By the tangent cone theorem)

Examples

•**1-formal groups**
Fundamental groups of compact Kähler manifolds, and fundamental groups of the complements of complex algebraic hypersurfaces.

•**Filtered-formal, but not graded-formal**
The upper triangular nilpotent group $U_n(\mathbb{Z})$, $n \geq 3$ is filtered-formal. The pure braid groups on surface $P_{n,1}$ for $n \geq 3$. Fundamental groups of Seifert fibered manifolds.

•**Graded-formal, but not filtered-formal**
The pure virtual braid groups vP_n and their subgroup vP_n^+ for $n \geq 4$.

•**Neither graded-formal nor filtered-formal**
If G_1 is not graded-formal and G_2 is not filtered-formal, then Theorem 2 shows that $G_1 \times G_2$ and $G_1 * G_2$ are neither graded-formal, nor filtered-formal.

Remark: (Resonance and Chen ranks)
The fundamental groups of the spaces of all configurations of parallel rings are also known as the pure welded braid groups, or the McCool groups, denoted by wP_n with a class of subgroups wP_n^+.
Berceanu and Papadima proved that the McCool groups and their upper subgroups are 1-formal. D. Cohen computed the first resonance varieties, while Cohen and Schenck showed that the Chen ranks of wP_n are given by the 'Chen ranks formula'.
In [3], we compute the first resonance varieties and the Chen ranks of wP_n^+ and show that the 'Chen ranks formula' hypothesis is not satisfied for wP_n^+.

Papers

[1] Alexander I. Suciu and He Wang.
Formality properties of finitely generated groups and lie algebras.
arXiv:1504.08294v1, April 2015.

[2] Alexander I. Suciu and He Wang.
Pure virtual braids, resonance, and formality.
Preprint, July 2015.

[3] Alexander I. Suciu and He Wang.
Basis-conjugating automorphisms of a free group: resonance varieties and chen ranks.
Preprint, July 2015.

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