Resonance varieties, Hilbert series and Chen ranks

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Overview

1. Cohomology jump loci
   - The resonance varieties
   - The characteristic varieties
2. Alexander Modules and Chen Lie algebras
3. McCool groups
4. Picture groups

The resonance varieties

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Definition

The *resonance varieties* of G are the homogeneous subvarieties of A^1

$$\mathcal{R}^i_d(G, \mathbb{C}) = \{a \in A^1 \mid \dim_{\mathbb{C}} H^i(A; a) \geq d\},$$

defined for all integers $i \geq 1$ and $d \geq 1$.

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- $\mathcal{R}^1_1(\mathbb{Z}^n, \mathbb{C}) = \{0\}$; $\mathcal{R}^1_1(\pi_1(\Sigma_g), \mathbb{C}) = \mathbb{C}^{2g}$, $g \geq 2$.

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- The *character variety* $\mathbb{T}(X) := \mathrm{Hom}(G, \mathbb{C}^*) = \mathrm{Hom}(G_{\mathrm{ab}}, \mathbb{C}^*)$ is an algebraic group, with multiplication $f_1 \circ f_2(g) = f_1(g)f_2(g)$ and identity $\mathrm{id}(g) = 1$ for $g \in G$ and $f_i \in \mathrm{Hom}(G, \mathbb{C}^*)$.

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- The *rank 1 local system* on X is a 1-dimensional $\mathbb{C}$-vector space $\mathbb{C}_\rho$ with a right $\mathbb{C}G$-module structure $\mathbb{C}_\rho \times G \to \mathbb{C}_\rho$ given by $\rho(g) \cdot a$ for $a \in \mathbb{C}_\rho$ and $g \in G$ for $\rho \in \mathbb{T}(X)$.

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The *characteristic varieties* of X over $\mathbb{C}$ are the Zariski closed subsets

$$\mathcal{V}^i_d(X, \mathbb{C}) = \{\rho \in \mathbb{T}(X) = \mathrm{Hom}(G, \mathbb{C}^*) \mid \dim_{\mathbb{C}} H_i(X, \mathbb{C}_\rho) \geq d\}$$

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- $\mathcal{V}^1_1(T^n, \mathbb{C}) = \{1\}$; $\mathcal{V}^1_1(\Sigma_g, \mathbb{C}) = (\mathbb{C}^*)^{2g}$ for $g \geq 2$.

Tangent Cone Theorem

Theorem (Dimca, Papadima, Suciu 09)

If G is 1-formal, then the tangent cone $\mathrm{TC}_1(\mathcal{V}^1_d(G,\mathbb{C}))$ equals $\mathcal{R}^1_d(G,\mathbb{C})$. Moreover, $\mathcal{R}^1_d(G,\mathbb{C})$ is a union of rationally defined linear subspaces of $H^1(G,\mathbb{C})$.

Alexander invariant

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$$0 \to G'/G'' \to G/G'' \to G/G' \to 0.$$

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Proposition (Hironaka(97), Libgober(98) ...)

$\mathcal{V}^1_d(G, \mathbb{C}) = V(E_{d-1}(B(G) \otimes \mathbb{C}))$ *for* $d \geq 1$.

Chen Lie algebras

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- $\theta_k(F_n) = (k-1)\binom{n+k-2}{k}$, $k \geq 2$. [Chen (51)]

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Corollary

$$\text{Hilb}(B(G) \otimes \mathbb{C}, t) = \sum_{k \geq 0} \theta_{k+2}(G) t^k.$$

McCool groups (pure welded braid groups) (group of loops)

- The McCool group wP_n is the group of basis-conjugating automorphisms, which is a subgroup of
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Theorem (D.Cohen (09))

The first resonance variety of McCool group wP_n is

$$\mathcal{R}^1_1(wP_n, \mathbb{C}) = \bigcup_{1\leq i<j\leq n} C_{ij} \cup \bigcup_{1\leq i<j<k\leq n} C_{ijk},$$

where $C_{ij} = \mathbb{C}^2$ and $C_{ijk} = \mathbb{C}^3$.

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Theorem (Suciu, W. (15))

The Chen ranks θ_k of wP^+ are given by $\theta_1 = \binom{n}{2}, \theta_2 = \binom{n}{3}, \theta_3 = 2\binom{n+1}{4}$,

$$\theta_k = \binom{n+k-2}{k+1} + \theta_{k-1} = \sum_{i=3}^{k} \binom{n+i-2}{i+1} + \binom{n+1}{4}, \; k \geq 4.$$

Corollary

The pure braid group P_n, the upper McCool group $P\Sigma_n^+$, and the product group $\Pi_n := \prod_{i=1}^{n-1} F_i$ are not *isomorphic for $n \geq 4$.*

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Proof:

$$\theta_4(P_n) = 3\binom{n+1}{4}, \theta_4(P\Sigma_n^+) = 2\binom{n+1}{4} + \binom{n+2}{5}, \theta_4(\Pi_n) = 3\binom{n+2}{5}.$$

The Chen ranks of P_n and Π_n were computed by D. Cohen and Suciu (95).

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Theorem (Suciu, W. (15))

The first resonance variety of upper McCool group wP_n^+ is

$$\mathcal{R}_1^1(wP_n^+, \mathbb{C}) = \bigcup_{1 \leq i < j \leq n-1} C_{i,j},$$

where $C_{i,j} = \mathbb{C}^{j+1}$.

Remark

There is a close connection (under some conditions) between the Chen ranks $\theta_k(G)$ and the resonance varieties $\mathcal{R}^1_1(G)$:

$$\theta_k(G) = \sum_{n\geq 2} c_m \cdot \theta_k(F_n), \quad \text{for} \quad k \gg 1,$$

where c_n is the number of n-dimensional components of $\mathcal{R}^1_1(G)$. (Suciu(01)) (Schenck and Suciu(04)) (D. Cohen and Schenck (14))

The pure braid groups P_n, the McCool groups wP_n, satisfy this formula. However, the upper McCool groups wP_n^+ does not satisfies this formula for $n \geq 4$.

Picture groups

- For every quiver of finite type there is a finitely presented group called picture group (Igusa, Orr, Todorov and Weyman (14)).

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  $$\begin{cases} (x_{ij}, x_{kl}) = 1, & \text{if } (i,j),(k,l) \text{ are noncrossing;} \\ (x_{ij}, x_{jk}) = x_{ik}, & \text{if } i < j < k, \end{cases}$$
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Lemma

There exists a surjection $R(A_n) \twoheadrightarrow G(A_n)$ inducing isomorphism on the resonance varieties $\mathcal{R}^1_d(G(A_n)) = \mathcal{R}^1_d(R(A_n))$.

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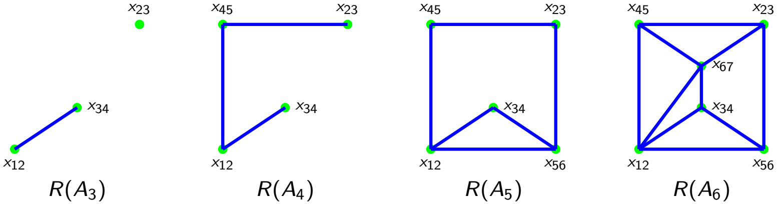

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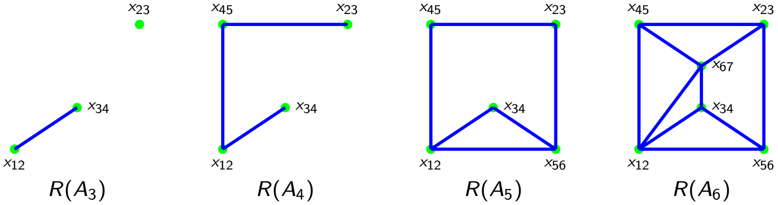

- All resonance varieties and characteristic varieties of right-angled Artin groups were computed by Papadima and Suciu (09). We only review the first resonance varieties here.

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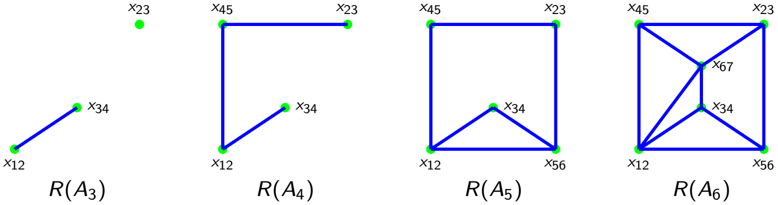

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Theorem (Papadima-Suciu (06))

Let $\Gamma = (V, E)$ be a finite graph. Then $\mathcal{R}_1^1(G_\Gamma; \mathbb{C}) = \bigcup_W \mathbb{C}^W$, where the union is over all subsets $W \subset V$ such that the induced subgraph Γ_W is disconnected. Here, $\mathbb{C}^W$ is the corresponding coordinate subspace of $\mathbb{C}^V$.

Corollary

Recall that the graph corresponding to $R(A_n)$ *has vertex set* $\{x_{i,i+1}, (1 \le i \le n)\}$ *and edges* $(x_{i,i+1}, x_{j,j+1})$ *for* $i < j - 1$.

$$\mathcal{R}_1^1(G(A_n)) = \mathcal{R}_1^1(R(A_n)) = \bigcup_{i=1}^{n-2} \mathbb{C}^{W_i}$$

where $W_i = \{x_{i,i+1}, x_{i+1,i+2}, x_{i+2,i+3}\}$.

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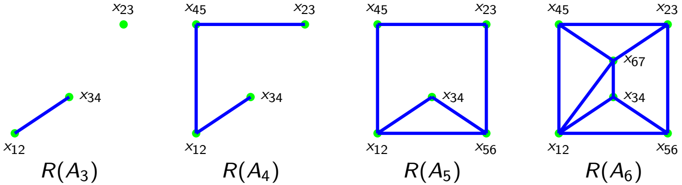

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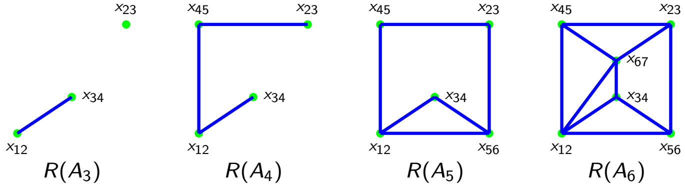

Example

$\mathcal{R}^1_1(G(A_3)) = \mathbb{C}^3 = H^1(G(A_3);\mathbb{C})$.
$\mathcal{R}^1_1(G(A_4)) = \mathbb{C}^3 \cup \mathbb{C}^3 \subset H^1(G(A_4);\mathbb{C}) = \mathbb{C}^4$.
$\mathcal{R}^1_1(G(A_5)) = \mathbb{C}^3 \cup \mathbb{C}^3 \cup \mathbb{C}^3 \subset H^1(G(A_5);\mathbb{C}) = \mathbb{C}^5$.
$\mathcal{R}^1_1(G(A_6)) = \mathbb{C}^3 \cup \mathbb{C}^3 \cup \mathbb{C}^3 \cup \mathbb{C}^3 \subset H^1(G(A_6);\mathbb{C}) = \mathbb{C}^6$.

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Thank You!