Cohomology jump loci of configuration spaces

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Overview

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- The $\mathbb{Z}[G_{\mathrm{ab}}]$-module structure on $B(G)$ is determined by the extension

$$0 \to G'/G'' \to G/G'' \to G/G' \to 0.$$

  with G/G' acting on the cosets of G'' via conjugation: $gG' \cdot hG'' = ghg^{-1}G''$, for $g \in G$, $h \in G'$.

Chen Lie algebra

- The *lower central series* G: $\Gamma_1 G = G$, $\Gamma_{k+1} G = [\Gamma_k G, G]$, $k \geq 1$.
- The *Chen Lie algebra* of a group G is defined to be

$$\mathrm{gr}(G/G''; \Bbbk) := \bigoplus_{k \geq 1} (\Gamma_k(G/G'')/\Gamma_{k+1}(G/G'')) \otimes_{\mathbb{Z}} \Bbbk.$$

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Proposition (Massey 80)

For each $k \geq 2$, there exists an isomorphism

$$\mathrm{gr}_k(G/G'') \cong \mathrm{gr}_{k-2}(B(G)).$$

Alexander varieties

Definition (Libgober 1992)

The *Alexander variety* of X (over $\mathbb{C}$)

$$\mathcal{W}^i_d(X, \mathbb{C}) = V(E_{d-1}(H_i(X^{\mathrm{ab}}, \mathbb{C})))$$

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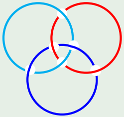
- $\mathcal{W}_d^1(\Sigma_g,\mathbb{C}) = (\mathbb{C}^*)^{2g}$ for $g > 1$, $d < 2g-1$.

Example (Borromean rings)

Let X be the complement in $\mathbb{S}^3$ of the Borromean rings:



A presentation for the fundamental group

$$G = \pi_1(X) = \langle x, y, z \mid zyz^{-1}xzy^{-1}z^{-1} = yxy^{-1}, xzx^{-1}yxz^{-1}x = zyz^{-1} \rangle.$$

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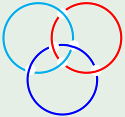

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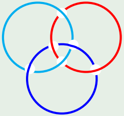

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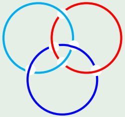

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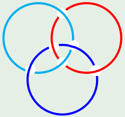

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- The Alexander variety
  $\mathcal{W}_1^1(X, \mathbb{C}) = \{t_1 = 1\} \cup \{t_2 = 1\} \cup \{t_3 = 1\} = (\mathbb{C}^*)^2 \cup (\mathbb{C}^*)^2 \cup (\mathbb{C}^*)^2$;
  $\mathcal{W}_2^1(X, \mathbb{C}) = \{t_1 = t_2 = 1\} \cup \{t_2 = t_3 = 1\} \cup \{t_3 = t_1 = 1\}$;
  $\mathcal{W}_3^1(X, \mathbb{C}) = \{1\}$.

The characteristic varieties

- The *rank 1 local system* on X is a 1-dimensional $\mathbb{C}$-vector space $\mathbb{C}_\rho$ with a right $\mathbb{C}G$-module structure $\mathbb{C}_\rho \times G \to \mathbb{C}_\rho$ given by $\rho(g) \cdot a$ for $a \in \mathbb{C}_\rho$ and $g \in G$ for $\rho \in \mathrm{Hom}(G, \mathbb{C}^*)$.

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Proposition (Papadima,Suciu10)

$$\bigcup_{i=0}^{q} \mathcal{V}^i_1(X, \mathbb{C}) = \bigcup_{i=0}^{q} \mathcal{W}^i_1(X, \mathbb{C}).$$

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Theorem (Dimca, Papadima, Suciu 09)

If G is 1-formal, then the tangent cone $\mathrm{TC}_1(\mathcal{V}^1_d(G, \mathbb{C}))$ equals $\mathcal{R}^1_d(G, \mathbb{C})$. Moreover, $\mathcal{R}^1_d(G, \mathbb{C})$ is a union of rationally defined linear subspaces of $H^1(G, \mathbb{C})$.

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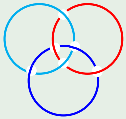

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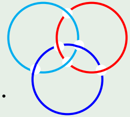

1-Formality and Tangent Cone Theorem

- A space X is *1-formal* if there exists a cdga morphism from the minimal model $\mathcal{M}(X)$ to $(H^*(X,\mathbb{Q}),0)$ inducing isomorphism in cohomology of degree 1 and monomorphism in degree 2.
- A group G is *1-formal* if the Eilenberg-MacLane space $K(G,1)$ is 1-formal.

Theorem (Dimca, Papadima, Suciu 09)

If G is 1-formal, then the tangent cone $\mathrm{TC}_1(\mathcal{V}^1_d(G,\mathbb{C}))$ equals $\mathcal{R}^1_d(G,\mathbb{C})$. Moreover, $\mathcal{R}^1_d(G,\mathbb{C})$ is a union of rationally defined linear subspaces of $H^1(G,\mathbb{C})$.

Example (Borromean link again)

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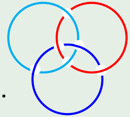
  $\Rightarrow X$ is not 1-formal.



The configuration spaces

Let M be a connected manifold with $\dim_{\mathbb{R}} M \geq 2$. The configuration space

$$\mathcal{F}(M, n) = \{(x_1, \cdots, x_n) \in M \times \cdots \times M \mid x_i \neq x_j \text{ for } i \neq j\}.$$

There is a free action of S_n on $\mathcal{F}(M, n)$ by permutation of coordinates, with orbit space $\mathcal{C}(M, n) = \mathcal{F}(M, n)/S_n$.

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Proposition (Cohen, Suciu 95)

The Chen ranks of P_n are given by

$$\theta_1(P_n) = \binom{n}{2};\ \theta_2(P_n) = \binom{n}{3};\ \theta_k(P_n) = (k-1)\binom{n+1}{4},\ \textit{for } k \geq 3$$

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Corollary

P_n is not isomorphic to $\Pi_n = F_1 \times \cdots \times F_{n-1}$ for $n \geq 4$.

The pure braid groups on Riemann surface

- $P_{g,n} = \pi_1(\mathcal{F}(\Sigma_g, n))$, where $\mathcal{F}(\Sigma_g, n)$ is the configuration of Σ_g, which is a smooth compact complex curve of genus g $(g \geq 1)$.

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$P_{n,1}$ is not 1-formal for $n \geq 3$.

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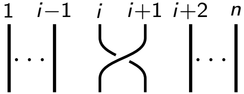

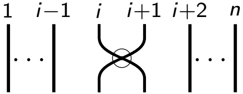

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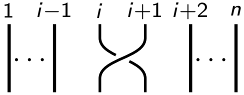

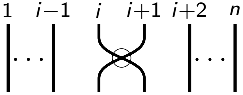

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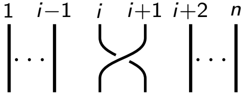

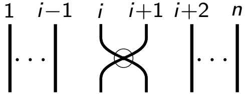

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- The pure virtual braid groups vP_n has presentation [Bardakov04]

$$\left\langle x_{ij}, (1 \leq i \neq j \leq n) \middle| \begin{array}{l} x_{ij}x_{ik}x_{jk} = x_{jk}x_{ik}x_{ij}; \\ x_{ij}x_{kl} = x_{kl}x_{ij}; i,j,k,l \text{ distinct} \end{array} \right\rangle.$$

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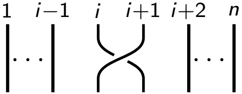

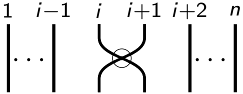

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- vP_n^+ is the quotient of vP_n by the relations $x_{ij}x_{ji} = 1$ for $i \neq j$.

Theorem (Suciu, W. 15)

The pure virtual braid groups vP_n and vP_n^+ are 1-formal if and only if $n \leq 3$.

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Sketch of proof:

Lemma

There are split monomorphisms

$$\begin{array}{ccccccccccc}
vP_2^+ & \hookrightarrow & vP_3^+ & \hookrightarrow & \color{red}{vP_4^+} & \hookrightarrow & vP_5^+ & \hookrightarrow & vP_6^+ & \hookrightarrow & \dots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
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Lemma

Suppose there is a split monomorphism $\iota \colon N \hookrightarrow G$.
If G is 1-formal, then N is also 1-formal.

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The first resonance variety $\mathcal{R}_1^1(vP_4^+, \mathbb{C})$ is the subvariety of $\mathbb{C}^6$ given by the equations

$$x_{12}x_{24}(x_{13}+x_{23}) + x_{13}x_{34}(x_{12}-x_{23}) - x_{24}x_{34}(x_{12}+x_{13}) = 0,$$
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$\Rightarrow$ The group vP_4^+ is not 1-formal.

The pure welded braid groups (McCool groups)

- The welded braid group wB_n has the same generators as vB_n, adding one more class of relations

$$\sigma_i\sigma_{i+1}s_i = s_{i+1}\sigma_i\sigma_{i+1}, i = 1, 2, \ldots, n-2.$$

- $1 \to wP_n \to wB_n \xrightarrow{\rho} S_n \to 1.$
- The pure welded braid groups wP_n has presentation [McCool 86]

$$\left\langle x_{ij}, (1 \leq i \neq j \leq n) \;\middle|\; \begin{array}{l} x_{ij}x_{ik}x_{jk} = x_{jk}x_{ik}x_{ij}; \\ x_{ij}x_{kl} = x_{kl}x_{ij}; i, j, k, l \text{ distinct} \\ x_{ij}x_{kj} = x_{kj}x_{ij}; i, j, k \text{ distinct} \end{array} \right\rangle.$$

- There is a subgroup of wP_n generated by the x_{ij} for $1 \leq i < j \leq n$, denoted by wP_n^+. The group wP_n is called McCool group and wP_n^+ is called upper McCool group.

Theorem (D.Cohen 09)

The first resonance variety of McCool group wP_n is

$$\mathcal{R}^1_1(wP_n, \mathbb{C}) = \bigcup_{1 \leq i < j \leq n} C_{ij} \cup \bigcup_{1 \leq i < j < k \leq n} C_{ijk},$$

where $C_{ij} = \mathbb{C}^2$ and $C_{ijk} = \mathbb{C}^3$.

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Theorem (Suciu, W. 15)

The first resonance variety of upper McCool group wP_n^+ is

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where $C_{i,j} = \mathbb{C}^{j+1}$.

Future work

- The relations between the Chen ranks $\theta_k(G)$ and $\mathcal{R}^1_1(G)$

$$\theta_k(G) = \sum_{m\geq 2} c_m \cdot \theta_k(F_m)$$

where c_m is the number of m-dimensional components of $\mathcal{R}^1_1(G)$. (Schenck and Suciu04) (Cohen and Schenck14)

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